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Estimates For Oscillator Integrals With A Special Phase

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ABSTRACT

In this paper, uniform estimates are considered for oscillatory integrals with some phase functions depending on small parameters.

KEYWORDS

Phase, Amplitude, Height, Deformation, Ideal.

INTRODUCTION

Let $f(x_1, x_2)$ $-$ be an infinitely smooth function that has a singularity at the point $(0,0)$, i.e. $df(0,0) = 0$. Consider a deformation of the phase function of the form

$$F(x,s) = f(x) + s_1\ell_1(x) + s_2\ell_2(x),$$

where $\ell_1, \ell_2 -$ smooth functions satisfying the conditions:

$$\ell_1(0,0) = 0, \ \ \ell_2(0,0) = 0, \ \ J(\ell_1, \ell_2) \neq 0 \qquad (1)$$

where J $-$jacobian of functions ℓ_1, ℓ_2. Let U be a neighborhood of zero and $a \in C_0^\infty(U)$. We introduce an oscillatory integral with phase $F(x,s)$ and amplitude $a \in C_0^\infty(U)$:

$$J(t,s) = \iint_{R^2} a(x)e^{itF(x,s)}dx \qquad (2)$$

An oscillatory integral with a smooth phase $f(x)$ is said to have an estimate of the type (β, m), at the point (0,0) if there exists a neighborhood U of zero such that for any function $a \in C_0^\infty(U)$ the estimate holds for $|t| \geq 2$

$$\iint_{R^2} a(x)e^{itF(x,s)}dx \leq C \cdot |lnt|^m \cdot |t|^{-\beta}.$$

The oscillation exponent of a function f at zero is called the supremum of the set $\{\beta\}$.

Theorem. Let $f(x)$ have an estimate of the type $(\beta, m), \beta \geq \frac{1}{2}$ at the point (0,0), and $\ell_1, \ell_2 -$ are any fixed functions satisfying conditions (1). Then there exists a neighborhood U of zero in R^2 and a positive number $\varepsilon > 0$ such that for the amplitude $a \in C_0^\infty(U)$ and $|s| < \varepsilon$, the oscillator integral (2) satisfies the following estimate:

$|J(t,s)| \leq C \cdot \|a\|_{C^2} \cdot |lnt|^m \cdot |t|^{-\beta}$.

Let $f(x_1, x_2) -$ be an infinitely smooth function with a singularity $f(0,0) = 0$, $d^2f(0,0) = 0$, $d^3f(0,0) = 0$ and $h(f) = 2$.The concepts of height and the adapted coordinate systems for the function were introduced by A.N.Varchenko in [2]. In this case $d^4f(0,0) \neq 0$. Since $h(f) = 2$, then either the principal face, up to linear equivalence, has the form $f_\gamma = x_1^4 + \alpha x_1^2x_2^2 + x_2^4$ and $a^2 \neq 4$, or f_γ is reduced to one of the forms $x_1^2x_2^2,\ x_1^2(x_1^2 \pm x_2^2), (x_1^2 + x_2^2)^2$.

As is known, if the principal face has the form $x_1^4 + \alpha x_1^2x_2^2 + x_2^4$ and $a^2 \neq 4$, then the phase function f in some neighborhood of zero is reduced by a differomorphism to the form $x_1^4 + \alpha x_1^2x_2^2 + x_2^4$. This feature is called the X_9 type feature [1]. In this case, the phase function $F(x,s)$ is reduced to normal form and the proof of the theorem follows from Karpushkin's theorem [4].

It remains to consider the cases when $f_\gamma = \pm x_1^2x_2^2$, $f_\gamma = x_1^2(x_1^2 \pm x_2^2)$ and $f_\gamma = (x_1^2 + x_2^2)^2$. For the sake of simplicity, we'll assume $f_\gamma = \pm x_1^2x_2^2$.

Lemma. If a function f at a point (0,0) is diffeomorphically equivalent $\pm x_1^2x_2^2$, then there exists a neighborhood U and a positive number $\varepsilon > 0$ such that for any amplitude $a \in C_0^\infty(U)$ and $|s| < \varepsilon$ the oscillator integral $J(t,s)$ satisfies the estimate:

$$|J(t,s)| \leq C \cdot \|a\|_{C^2} \cdot |lnt| \cdot |t|^{-1/2}.$$

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Proof. In this case, we represent the phase function in the form

$$F(x,s) = x_1^2x_2^2 + s_1(x_1 + \varphi_1(x_1,x_2)) + s_2(x_2 + \varphi_2(x_1,x_2)),$$

where $\varphi_k \in \mathcal{M},\ k = 1, 2$ are smooth functions. In what follows, $\mathcal{M}$ will denote the maximal ideal of the ring of germs of smooth functions.

In this case, (s_1, s_2) and (x_1, x_2) are symmetric. Therefore, we will consider only the case $(s_1, s_2) \in \{(s_1, s_2): |s_2| \le |s_1|\}$. The phase function $F(x,s)$ can be represented in the form

$$F(x,s) = x_1^2x_2^2 +$$
$$+s_1x_1[1 + \varphi_{11}(x_1,x_2) + \xi_2\varphi_{21}(x_1,x_2) + s_1x_2^2\varphi_{12}(x_2) + s_2(x_2 + x_2^2\varphi_{22}(x_2))]$$

where $\varphi_{11}, \varphi_{21} \in \mathcal{M}$ and φ_{12}, φ_{22} are some smooth functions. Consider the one-dimensional oscillatory integral

$$J_1(t,s,x_2) = \int_{R_{x_1}} \exp\left\{itx_2^2\left(x_1^2 + \frac{s_1}{x_2^2}x_1\big(1 + \varphi_{11}(x_1,x_2) + \xi_2\varphi_{21}(x_1,x_2)\big)\right)\right\} a(x_1,x_2)dx_1$$

Let $|s_1| > \delta|x_2^2| > 0$ where δ – is some fixed positive number. Then the oscillator integral $J_1(t,s,x_2)$ satisfies the estimate:

$$J_1(t,s,x_2) \le \frac{C\|a\|_{C^1}}{1+|tx_2^2|}.$$

From here we get:

$$\int\limits_{|s_1|>\delta|x_2^2|} |J_1(t,s,x_2)|\,dx_2 \le \frac{C\|a\|_{C^1}}{|t|^{1/2}}.$$

Let $|s_1| \le \delta|x_2^2|$ where δ ‑ is a sufficiently small positive number. In this case, according to the van der Corput lemma [5], the integral $J_1(t,s,x_2)$ satisfies the inequality

$$|J_1(t,s,x_2)| \le \frac{C\|a\|_{C^1}}{1+|tx_2^2|^{1/2}}.$$

As a result, we have

$$\int\limits_{|s_1|\le\delta|x_2^2|\le C_1} |J_1(t,s,x_2)|\,dx_2 = C\|a\|_{C^1}\int\limits_{|x_2|\le C}\frac{dx_2}{1+|tx_2^2|^{1/2}} \le \frac{C\|a\|_{C^1}|lnt|}{|t|^{1/2}}.$$

The proof of the lemma follows easily from this.

This lemma implies the proof of the theorem in the case when f at the point (0,0) is diffeomorphically equivalent to the function $\pm x_1^2 x_2^2$.

Finally, consider the case when the “almost” principal face (in the terminology of Karpushkin [4]) has the form:

$$f_\gamma(x_1,x_2) = x_1^2 x_2^2 + a x_1^k, \quad (k \geq 4).$$

Note that we always arrive at this case by changing the variables. First, assume that $|s_2| \leq |s_1|$. We represent the phase function in the form

$$F(x,s) = x_1^2 x_2^2 + a x_1^k + F_{1>}(x_1,x_2) + s_1 x_2^2 \varphi_{11}(x_2) +$$
$$+ s_1 x_1 [1 + \varphi_{11}(x_1,x_2) + \xi_2 \varphi_{21}(x_1,x_2)] + s_2 x_2^2 \varphi_{22}(x_2).$$

Without loss of generality, we can assume that $a \neq 0$, otherwise either it is reduced to this case, or the phase function is diffeomorphically equivalent to the function $\pm x_1^2 x_2^2$. Let $x_2 > 0$, consider the one-dimensional oscillatory integral

$$J_1(t,s,x_2) = \int_{R_{x_1}} e^{\Phi(x,s)} a(x_1,x_2) dx_1,$$

where

$$\Phi(x,s) = x_1^2 x_2^2 + a x_1^k + F_{1>}(x_1,x_2) + s_1 x_1 [1 + \varphi_{11}(x_1,x_2) + \xi_2 \varphi_{21}(x_1,x_2)]$$

here $F_{1>}$ the Maclaurin series of the function consists of the sum of monomials of degree higher than one with weight $r = \left(\frac{1}{k}, \frac{k-2}{2k}\right)$, $\varphi_{11}, \varphi_{21} \in \mathcal{M}$.

Let's make the change of variables $x_1 \longmapsto x_2^{\frac{2}{k-2}} x_1$ and get:

$$J_1(t,s,x_2) = \int_{R_{x_1}} x_2^{\frac{2}{k-2}} e^{itx_2^{\frac{2k}{k-2}} \Phi_1(x_1,x_2,s)} a\left(x_2^{\frac{2}{k-2}} x_1, x_2\right) dx_1,$$

where $\Phi_1(x_1,x_2,s) = a x_1^k + x_1^2 + s_1 x_2^{-\frac{2(k-1)}{k-2}} x_1 \left[1 + \varphi_{11}(x_2^{\frac{2}{k-2}} x_1, x_2) + \xi_2 \varphi_{21}(x_2^{\frac{2}{k-2}} x_1, x_2)\right]$

First, consider the case $|s_1 x_2^{-\frac{2(k-1)}{k-2}}| \leq M$, where M – is a fixed positive number. The set of critical points of the phase function $\Phi_1(x_1,x_2,s)$ in x_1 is contained on some interval [-Δ, Δ]. Consider the covering $(-\Delta - 1, \ \Delta + 1) \ \cup \ (R \backslash [-\Delta, \Delta])$ of the set R, and denote the corresponding partition of unity by $\{h_1, h_2\}$. With this partition of unity, the oscillatory integral $J_1(t,s,x_2)$ is represented as the sum of two integrals

$$J_{1k}(t,s,x_2) = x_2^{\frac{2}{k-2}} \int_R e^{itx_2^{\frac{2k}{k-2}}\Phi_1(x_1,x_2,s)} a\left(x_2^{\frac{2}{k-2}}x_1, x_2\right) h_k(x_1)dx_1, \;\; k = 1,2.$$

Applying van der Corput's lemma for the oscillatory integral $J12$, we obtain the estimate:

$$J_{12}(t,s,x_2) \leq \frac{C\|a(\cdot,x_2)\|_V}{\left|tx_2^{\frac{2(k-1)}{k-2}}\right| + |t|^{\frac{1}{k}}}.$$

From the latter we have

$$\int_R |J_{12}(t,s,x_2)|dx_2 \leq C \cdot \|a\|_{C^2} \cdot |t|^{-\frac{1}{2}}$$

Now consider the estimate for the oscillatory integral $J_{11}(t,s,x_2)$. Note that there is at most one point, $\xi_1 = \xi_0 \neq 0$ for which the function $ax_1^k + x_1^2 + \xi_0 x_1$ has a degenerate critical point. If $k -$ is even and a is positive, then there is no such point. Let such a point exist. Then the phase function is a versal deformation of a singularity of type A_2, and for $|\xi_1 - \xi_0| < \delta$ for the oscillatory integral $J_1(t,s,x_2)$, the estimate [3] holds.

$$|J_1(t,s,x_2)| \leq \frac{C \cdot \|a(\cdot,x_2)\|_V}{|t|^{\frac{1}{2}}|x_2|\left|s_1 x_2^{-\frac{2(k-1)}{k-2}} - \xi_0\right|^{\frac{1}{4}}} := \psi_1(s_1,x_2,t).$$

If $|\xi_1 - \xi_0| > \delta$, then all critical points of the phase function are nondegenerate and the oscillatory integral $J_1(t,s,x_2)$ satisfies the estimate:

$$|J_1(t,s,x_2)| \leq \frac{C \cdot |x_2|^{\frac{2}{k-2}} \cdot \|a(\cdot,x_2)\|_V}{1 + \left|tx_2^{\frac{2k}{k-2}}\right|^{\frac{1}{2}}} := \psi_2(x_2,t).$$

Note that the following inequalities hold:

$$\int_{|\xi_1-\xi_0|<\delta} \psi_1(s_1,x_2,t)dx_2 \leq C \cdot \|a\|_{C^2} \cdot |lnt| \cdot |t|^{-\frac{1}{2}},$$

$$\int_{|\xi_1-\xi_0|>\delta} \psi_1(s_1,x_2,t)dx_2 \leq C \cdot \|a\|_{C^2} \cdot |t|^{-\frac{1}{2}}$$

Summing up the estimates obtained, we arrive at the desired estimate in the case | $|\xi_1| \leq M$. Now consider the case $|\xi_1| > M$ and M is a sufficiently large positive number. In this case, in the oscillator integral $J_1(t,s,x_2)$ we change the variables $x_1 \to |s_1|^{\frac{1}{k-1}}x_1$ and get:

$$|J_1(t,s,x_2)| = |s_1|^{\frac{1}{k-1}} \int\limits_R \exp\{it|s_1|^{\frac{1}{k-1}}\Phi_2(x,s)\}\, a\left(|s_1|^{\frac{1}{k-1}}x_1, x_2\right) dx_1,$$

where

$$\Phi_2(x,s) = ax_1^k + F_{1>}\left(|s_1|^{\frac{1}{k-1}}x_1, x_2\right)|s_1|^{-\frac{k}{k-1}} + |s_1|^{-\frac{k-2}{k-1}}x_1x_2^2 +$$

$$+sgns_1x_1\left[1+\varphi_{11}\left(|s_1|^{\frac{1}{k-1}}x_1, x_2\right) + \xi_2\varphi_{21}\left(|s_1|^{\frac{1}{k-1}}x_1, x_2\right)\right].$$

Let $\eta_2 = |s_1|^{-\frac{k-2}{k-1}}x_2^2$. Since $a \neq 0$, the set of critical points is contained in $[-\Delta, \Delta]$. Consider the covering $(-\Delta-1, \Delta+1) \cup (R\backslash[-\Delta,\Delta])$ and the corresponding partition of the unity $\{\psi_1,\psi_2\}$. With the help of this partition, the unit of the oscillatory integral $J_1(t,s,x_2)$ is represented as the sum of two integrals

$$J_1^k(t,s,x_2) = |s_1|^{\frac{1}{k-1}} \int\limits_R e^{it|s_1|^{\frac{k}{k-1}}\Phi_2(x,s)}\, a\left(|s_1|^{\frac{1}{k-1}}x_1, x_2\right)\psi_k(x_1)dx_1, k = 1,2$$

Consider the estimate $J_1^2(t,s,x_2)$. According to van der Corput's lemma, for this oscillatory integral we obtain the estimate:

$$\left|J_1^2(t,s,x_2)\right| \leq \frac{C\cdot\|a(\cdot,x_2)\|_V}{|t|^{\frac{1}{k}} + |t||x_2|^{\frac{2(k-1)}{k-2}}}$$

Hence,

$$\int\limits_{|x_2|<C} \left|J_1^2(t,s,x_2)\right| dx_2 \leq C\cdot\|a\|_{C^2}\cdot|t|^{-\frac{1}{2}}.$$

Note that if $|\eta_2| < \delta$ and δ is a sufficiently small positive number, then the phase function has only non-degenerate critical points. As a consequence, for the oscillatory integral $J_1^1(t,s,x_2)$ we have the estimate

$$\left|J_1^1(t,s,x_2)\right| \leq \frac{C\cdot\|a(\cdot,x_2)\|_V}{|t|^{\frac{1}{k}} + |t|\cdot|x_2|}$$

As a result, we get:

$$\int\limits_{|x_2|<C} \left|J_1^1(t,s,x_2)\right| dx_2 \leq C\cdot\|a\|_{C^2}\cdot|lnt|\cdot|t|^{-\frac{1}{2}}.$$

Summing up the obtained inequalities, we have the desired estimate when $f_\gamma(x_1, x_2) = x_1^2 x_2^2 + a x_1^k$ and $|s_2| \le |s_1|$. In case $|s_1| \le |s_2|$ the integrals are estimated in exactly the same way as this estimate. This completes the proof of the main theorem.

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